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Steady compressible swirl flows with closed streamlines at high Reynolds numbers

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In steady axisymmetric flows in a closed swirl chamber one can distinguish between the swirl flow proper, with components normal to the meridian plane, and a secondary flow whose components lie in the meridian plane. One can trace the motion of a particle within the meridian plane. The closed path so obtained will be called a streamline, to be parametrized by a stream function ψ. One can distinguish between the flow in a boundary layer, where the velocity gradient is large, and a core flow, where the velocity and temperature gradients are relatively small. The present article is concerned with only the core flow. In high Reynolds number flows in which the streamlines are not closed there are three quantities which are constant along the streamlines: the total enthalpy (the right-hand side of Bernoulli's equation), the entropy and the moment of momentum of the particles with respect to the axis of symmetry. These are determined by conditions in the entrance cross-section. In flows with closed streamlines these quantities are ultimately determined by the cumulative effects of viscosity and heat conductivity. Conditions expressing cumulative effects enter the analysis as integrability conditions necessary for the existence of a second approximation in a development of the flow field with respect to the reciprocal of the Reynolds number. They are re-expressed as the balance equations for energy, entropy and angular momentum which are to be satisfied on all surfaces ψ = constant. One thus obtains an algorithm which leads from expressions for the total enthalpy, the entropy and the angular momentum as functions of ψ to the residuals in the balance equations, also computed as functions of ψ. The functions with which this algorithm starts must be chosen such that the residuals in the balance equations become zero. The secondary flow can be arbitrarily slow only if the Prandtl number is $\frac{1}{2}$. At the centre of the secondary motion the balance equations are linearly dependent. This fact introduces an additional free parameter which allows one to compute secondary flows with different speeds. The linearized algorithm has the character of a Fredholm integral equation. This suggests an iterative solution similar to a Neumann series. The particles experience periodic changes of state which can be discussed as thermodynamic cycles. Such an analysis shows that the heat inputs occur on average at lower temperatures than the heat outputs. Besides the work that maintains the swirl motion, which is provided by shear-force components normal to the meridian plane, one therefore needs additional work, provided by the shear-force components within the meridian plane, to maintain a secondary motion. Responsible for this state of affairs is the fact that particles which do not quite maintain adiabatic temperatures move within a field with a large pressure gradient caused by the swirl component. This makes the flow sensitive to disturbances in the energy balance.

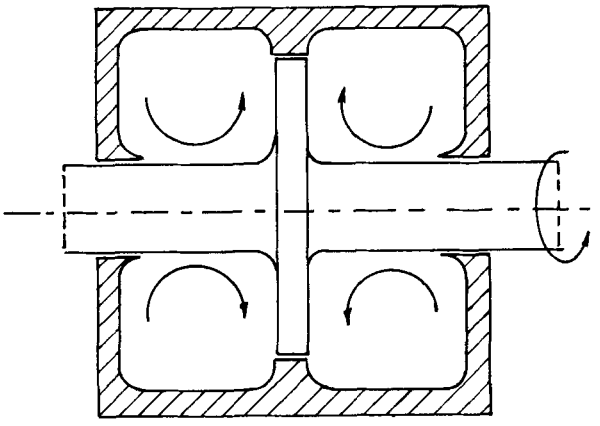


FIGURE 1. Prototype of a swirl chamber.

1. Introduction

Flow patterns of the kind investigated in the present article arise in the experimental arrangement shown in figure 1. The rotation of a disk (which is mounted on a relatively thick shaft) generates a velocity field with a rather large component, called the swirl component, perpendicular to the meridian plane. The effect of centrifugal forces in the boundary layer causes a secondary flow with velocity components in the meridian plane to arise. Flow patterns of a similar character can be obtained by tangential injection of gas at the surface of a chamber with fixed walls. In the interior of such 'swirl chambers' the velocity and temperature gradients are not very large; for high Reynolds numbers one can therefore compute the flow field from Euler's equations of motion, i.e. from equations in which viscosity and heat conductivity are neglected. The region where this simplification is permissible will be called the core of the flow. At the surface of a swirl chamber one has, of course, a boundary layer in which the gradients are large.

The fact that particles which belong to the core region stay there permanently has the consequence that cumulative effects of viscosity and heat conductivity will play a role in the final steady flow, even though the changes caused by them during a limited time are very small. For an incompressible flow this problem has been treated by Prandtl (1906) and by Batchelor (1956). Novel in the case of a compressible medium is the influence of the temperature field, which gives rise to physical phenomena not encountered before. The general state of affairs can be described as follows. The core flow is computed from equations in which viscosity and heat conductivity are neglected. One then finds that three quantities are constant along the streamlines, namely the entropy, the total enthalpy (i.e. the enthalpy at a fictitious stagnation point belonging to the streamline under consideration) and the moment of momentum of a particle with respect to the axis of symmetry. For streamlines which are not closed these quantities are determined by conditions at the entrance cross-section of the flow field, their changes due to viscosity and heat conduction as the particle passes through the flow field being neglible. For closed streamlines this information is, of course, not available; it is replaced by the requirement that the cumulative effects of viscosity and heat conduction must vanish. These conditions enter the present analysis as inte-

grability conditions for the second approximation in a development of the flow field with respect to the reciprocal of the Reynolds number. By a subsequent integration one obtains an even simpler requirement, namely that momentum balance, energy balance and entropy balance must be satisfied for each volume bounded by an axisymmetric surface ψ = constant.

This article outlines the important steps in the development of these relations. Moreover, it discusses certain properties of the resulting algorithm which are important for a general understanding as well as for the numerical work. Finally a number of examples are discussed. Qualitative insight regarding the properties of such flow fields can be obtained by analysing the cycles which the individual particles experience from a thermodynamic point of view. For the sake of brevity most of the details of the derivations have been omitted; these may be found in Guderley & Greene (1967), Guderley, Greene & Valentine (1975) and also in an earlier version of the present article (Guderley 1976).

2. General concepts

Let x and y respectively be co-ordinates in the direction of the axis of symmetry and in the radial direction of an axisymmetric flow field. Velocity components in the x and y directions are denoted by u and v. The angular velocity of the particles around the axis of symmetry is ω. Let $q^2 = u^2 + v^2$ and let p, ρ, i, s and T denote respectively pressure, density, specific enthalpy, specific entropy and absolute temperature. The coefficients of viscosity are μ and μ_1 and the coefficient of heat conduction k. Besides the equation of state of the gas, the flow is governed by the equations of conservation of mass, the Navier–Stokes equations and the equation of conservation of energy (here indirectly expressed in terms of the entropy changes along the path of a particle). Viscosity and heat-conduction effects, regarded as small, are relegated to the right-hand sides:

$$\partial(y\rho u)/\partial x + \partial(y\rho v)/\partial y = 0, \tag{2.1}$$

$$\rho^{-1}\partial p/\partial x + u\partial u/\partial x + v\partial u/\partial y = R_1, \tag{2.2}$$

$$\rho^{-1}\partial p/\partial y + u\partial v/\partial x + v\partial v/\partial y = R_2, \tag{2.3}$$

$$u\partial(y^2\omega)/\partial x + v\partial(y^2\omega)/\partial y = R_3, \tag{2.4}$$

$$u\partial s/\partial x + v\partial s/\partial y = R_4. \tag{2.5}$$

The functions R_1, R_2 and R_3 are the familiar viscosity terms of the Navier–Stokes equations written in cylindrical co-ordinates. R_4 is given by

$$R_4 = (\rho y T q)^{-1}[\partial(ykT_x)/\partial x + \partial(ykT_y)/\partial y + y\Phi], \tag{2.6}$$

where Φ is the dissipation function. Separating the dissipation due to the swirl motion from that due to the secondary motion, we write

$$\Phi = \Phi_1 + \Phi_2, \tag{2.7a}$$

where

$$\Phi_1 = \mu\{2[u_x^2 + v_y^2 + (v/y)^2] + [u_y + v_x]^2\} - \mu_1[u_x + v_y + (v/y)]^2, \tag{2.7b}$$

$$\Phi_2 = \mu y^2(\omega_x^2 + \omega_y^2). \tag{2.7c}$$

To satisfy (2.1), we introduce a stream function by setting

$$y\rho u = \psi_y, \quad -y\rho v = \psi_x. \tag{2.8}$$

For the flow patterns investigated here the particles move in three-dimensional space on toroidal surfaces given by ψ = constant. The intersection of such a surface with a meridian plane will be called a streamline. Let dl and dn be line elements in the direction of the streamline and in the direction of its normal. One then has

$$d(\,)/dl = -(u/q)\,\partial(\,)/\partial x - (v/q)\,\partial(\,)/\partial y,$$

$$d(\,)/dn = -(v/q)\,\partial(\,)/\partial x + (u/q)\,\partial(\,)/\partial y.$$

Using the relation $Tds = di - \rho^{-1}dp$, familiar from the thermodynamics, one finds from (2.1)–(2.5)

$$d(y^2\omega)/dl = -R_3, \quad ds/dl = -R_4,$$

$$d(i + \tfrac{1}{2}q^2 + \tfrac{1}{2}y^2\omega^2)/dl = -[(u/q)\,R_1 + (v/q)\,R_2 + \omega R_3 + TR_4].$$

The R_i vanish for zero viscosity and zero heat conductivity, in which case

$$(y^2\omega) = \hat{g}_1(\psi) = [g_1(\psi)]^{\frac{1}{2}}, \tag{2.9}$$

$$i + \tfrac{1}{2}q^2 + \tfrac{1}{2}y^2\omega^2 = g_2(\psi), \tag{2.10}$$

$$s = g_3(\psi). \tag{2.11}$$

The moment of momentum $y^2\omega$, the expression $i + \frac{1}{2}\,q^2 + \frac{1}{2}\,y^2\omega^2$ (which will be called the total specific enthalpy) and the entropy s are single-valued functions of x and y. After travelling around one streamline one finds oneself at the same point (x, y). Solutions to the steady equations (2.2)–(2.5) can therefore be found only if the functions R_i satisfy the integrability conditions

$$\oint R_3\,dl = 0, \quad \oint R_4\,dl = 0, \tag{2.12), (2.13}$$

$$\oint [(u/q)\,R_1 + (v/q)\,R_2 + \omega R_3 + TR_4]\,dl = 0, \tag{2.14}$$

where the paths of integration are any line ψ = constant.

One expects to be able to compute the flow field by means of a formal development with respect to the coefficients of viscosity and heat conductivity if the Reynolds number is large. Then one starts with the solutions of the inviscid equations. For this computation one needs the functions g_i, but at this stage they are arbitrary. If one prescribes these functions in some fashion, then the flow field is determined. If the coefficients of viscosity and heat conductivity are small then the R_i are small (provided of course that the derivatives which occur in the R_i are not large). The right-hand sides will therefore modify the inviscid flow by only small amounts. For the first approximation it is therefore permissible to compute the R_i from the inviscid approximation to the flow field. In the next approximation one would have the task of solving (2.2)–(2.5) again, but then the functions on the right appear; they are determined from the inviscid flow field. This presupposes that the integrability conditions (2.12)–(2.14) are satisfied. These equations require that the integration be carried out along the streamlines of the resulting flow field. It is shown in Guderley (1976) that in a consistent approximation the integrations can be carried out along the streamlines of the original

inviscid flow field. An inviscid flow field for which these integrability conditions are violated cannot be considered as a first approximation to a flow field in which a small viscosity and heat conductivity are present, for then one cannot find a correction which accounts for their effect. Accordingly one must choose the function g_i in such a manner that the integrability conditions are satisfied.

An interpretation of the integrability conditions as cumulative effects is not difficult. These conditions can be written in a form which has a different, more obvious, physical meaning. This is suggested by (2.12). The function R_3 gives the moment of the viscosity forces with respect to the axis of symmetry. This suggests that (2.12) is an expression of the law of moment of momentum applied to particles lying between the toroidal surfaces ψ = constant and $\psi + d\psi$ = constant. One expects, therefore, that one can derive from (2.12) the law of momentum for all particles that move within a surface ψ = constant. We proceed as follows. The integral $\oint R_3\,dl$ defines a function of ψ. We integrate this function with respect to ψ, the dummy variable of integration being denoted by ξ:

$$I_1(\psi) = \int_0^{\psi} \left(\oint_{\xi=\text{const}} R_3\,dl \right) d\xi.$$

The point of the flow field where the velocity components u and v are zero will be called the centre of the secondary motion. At this point we set $\psi = 0$. It follows from (2.8) that $d\psi/dn = y\rho q$ and therefore also $d\xi/dn = y\rho q$. Thus we write

$$I_1(\psi) = \int_0^{\psi} y\rho q \left(\oint_{\xi=\text{const}} R_3\,dl \right) dn.$$

The right-hand side can be considered as an area integral in the x, y plane, for $dn\,dl$ is an area element. Thus one has

$$I_1(\psi) = \iint_{A(\psi)} y\rho q R_3\,dx\,dy,$$

where $A(\psi)$ denotes the area enclosed by the line ψ = constant. The expression for R_3 is

$$R_3 = (y\rho q)^{-1}[\partial(y^3\mu\omega_x)/\partial x + \partial(y^3\mu\omega_y)/\partial y].$$

Therefore

$$I_1(\psi) = \iint_{A=(\psi)} [\partial(y^3\mu\omega_x)/\partial x + \partial(y^3\mu\omega_y)/\partial y]\,dx\,dy.$$

The integrand has a form which allows the application of Gauss' divergence theorem. Thus one obtains

$$I_1(\psi) = \oint_{\psi=\text{const}} \mu y^3(\omega_x\,dy - \omega_y\,dx). \tag{2.15}$$

The right-hand side indeed gives the moment with respect to the axis $y = 0$ of the viscosity forces exerted on the toroidal surface ψ = constant. The requirement $I_1 = 0$ must be satisfied for each value of ψ. The other integrability conditions are treated in an analogous manner. Defining

$$I_2(\psi) = \int_0^{\psi} \left(\oint_{\xi=\text{const}} R_4\,dl \right) d\xi,$$

substituting the definition (2.6) of R_4 and applying Gauss' theorem to part of the integrand, one obtains

$$I_2(\psi) = \oint_{\psi=\text{const}} (ky/T)\,[T_x\,dy - T_y\,dx] + \iint_{A(\psi)} [(ky/T^2)\,(T_x^2+T_y^2) + (y/T)\,\Phi]\,dx\,dy. \tag{2.16}$$

The line integral represents the entropy influx through the surface ψ = constant while the area integral gives the entropy generation in the interior, which is due to heat conduction and dissipation. In this case the occurrence of an area integral cannot be avoided; the introduction of I_2 is nevertheless advantageous because it eliminates third derivatives of ψ.

We define furthermore

$$I_3(\psi) = \int_0^{\psi} \left(\oint_{\xi=\text{const}} [(u/q)\,R_1 + (v/q)\,R_2 + \omega R_3 + TR_4])\,dl \right) d\xi.$$

Equation (2.14) is closely related to Bernoulli's equation, therefore one expects that I_3 will give an expression for the energy balance. It must therefore be possible to transform I_3 into a surface integral. Details may be found in Guderley & Greene (1967) and Guderley (1976). The part of the integrand in which R_1, R_2 and R_3 occur is not in a form which permits the immediate application of Gauss' theorem. Such a form is obtained if one splits off an expression which is the negative of the dissipation function (which appears in the contribution TR_4). The heat-conduction terms are already in a form which allows the application of Gauss' theorem. Ultimately one finds

$$\begin{aligned} I_3(\psi) = \oint_{\psi=\text{const}} \{&2\mu y(uu_x\,dy - vv_y\,dx) + \mu y(u_y+v_x)\,(v\,dy - u\,dx) \\ &+ \mu y^3\omega(\omega_x\,dy - \omega_y\,dx) + yk(T_x\,dy - T_y\,dx)\}. \end{aligned} \tag{2.17}$$

Now conditions (2.12)–(2.14) are replaced by

$$I_1(\psi) \equiv 0, \quad I_2(\psi) \equiv 0, \quad I_3(\psi) \equiv 0. \tag{2.18}$$

The physical meaning of these conditions is obvious. If the Navier–Stokes equations were exactly satisfied, then corresponding conditions would hold for any toroidal surface in the flow field. Under the present assumptions this requirement is weakened: only surfaces ψ = constant need be considered. One is tempted to take these conditions as the starting point of the theory, but then they would appear as arbitrary, although plausible, postulates, while in the present derivation they are a direct consequence of the integrability conditions for the system of perturbation equations.

3. Steady flow with zero viscosity and heat conductivity

The primary dependent variables in the subsequent development are the stream function ψ and the specific enthalpy i. The specific entropy is available as a function of ψ. It is therefore assumed that the thermodynamic state of the particles may be expressed in terms of s and i. One has, for instance, $\rho = \rho(i, s)$ and therefore

$$d\rho = (\partial\rho/\partial i)_s\,di + (\partial\rho/\partial s)_i\,ds.$$

We write in accordance with the usual definitions

$$(\partial\rho/\partial p)_s = 1/a^2.$$

Then, because of the second law of thermodynamics,

$$(\partial\rho/\partial i)_s = \rho/a^2.$$

Using the definition (2.8) of the stream function and also (2.10), one obtains

$$i + (\psi_x^2 + \psi_y^2/(2y^2\rho^2) = g_2(\psi) - g_1(\psi)/2y^2. \tag{3.1}$$

We define a Mach number of the secondary motion by

$$M^2 = q^2/a^2 = (\psi_x^2 + \psi_y^2)/(y^2\rho^2a^2). \tag{3.2}$$

The rather lengthy derivation of the differential equation for ψ can be found in Guderley (1976). The result assumes a familiar form, except for the presence of the functions g_i, which are usually constant:

$$\begin{aligned}\psi_{xx} + \psi_{yy} - [(u^2/a^2)\,\psi_{xx} + (2uv/a^2)\,\psi_{xy} + (v^2/a^2)\,\psi_{yy}] \\ = (\psi_y/y)\,(1 + g_1/y^2a^2) + y^2\rho^2[dg_2/d\psi - \tfrac{1}{2}y^{-2}dg_1/d\psi \\ + (dg_3/d\psi)\,[(\rho_s/\rho)\,q^2 - (1 - M^2)\,T].\end{aligned} \tag{3.3}$$

This equation, in conjunction with (3.1) and of course other equations which define auxiliary variables such as ρ, T, a^2, u, v, q and M^2 in terms of the enthalpy i, the entropy $s = g_3(\psi)$ and derivatives of ψ, constitutes the inviscid flow problem. At the outer edge of the core region (which presumably coincides with the outer edge of the swirl chamber) one has ψ = constant. This constant is not arbitrary; if the functions g_i are given, it is indirectly determined by the requirement that $\psi = 0$ at the centre of the secondary motion.

For $q < a$ the problem is elliptic; notice that the type of the differential equation does not depend upon the swirl component. In all our computations it is assumed that $q < a$. This restriction is not unrealistic, although the value of a^2 becomes small, even negative, if y becomes small. The present analysis cannot be carried out for swirl chambers which extend to small values of y.

4. Some properties of the algorithm

We begin with a relation which is an integral part of the mathematical formulation of the problem. Let a subscript m characterize quantities evaluated at the centre of the secondary motion. In appendix B of Guderley *et al.* (1975) and, specialized to an ideal gas, in Guderley (1976), the following relation has been derived:

$$-T_m I_2 + I_3 - (g_1^{\frac{1}{2}}/y^2)_m I_1 = O(\psi^2). \tag{4.1}$$

(The individual terms, for instance I_2 or I_3, are $O(\psi)$.)

According to this equation the three balance equations are linearly dependent at the centre of the secondary motion. To arrive at this result one must solve the equations (3.1) and (3.3) for inviscid flow in the vicinity of the centre of the secondary motion. Since at this point $u = 0$ and $v = 0$ one obtains Poisson's equation with a constant inhomogeneous term. In the limit $\psi \to 0$ the streamlines approach ellipses. One arrives at (4.1) by evaluating the expressions I_1, I_2 and I_3 for these streamlines to the lowest order in ψ.

This result is important for the following reason. Consider a swirl chamber with fixed geometry. There are a number of quantities that can be changed in an experiment.

One of them is obviously the strength of the swirl component. For problems where μ = constant this quantity is directly given by g_1 and is easily changed in the computations. In addition one can change the average temperature and the average pressure in the swirl chamber. These two quantities are controlled in the equations by the choice of a reference state. This becomes most obvious if one considers an ideal gas and works with dimensionless quantities. One expects in addition that, within certain limits, the speed of the secondary motion can be controlled, for instance by changing the conditions in the boundary layer of the swirl chamber. (The control is not complete; we shall find that for $Pr \neq \frac{1}{2}$ the secondary motion cannot be arbitrarily slow.) It is therefore likely that, for a fixed geometry, a fixed reference swirl and a fixed thermodynamic reference state, one can still find a continuous one-parameter family of solutions. This additional arbitrariness is due to the relation (4.1). This can be seen by the following heuristic argument. The difference between two adjacent solutions of this family (technically speaking, the derivatives of the flow quantities with respect to the parameter which characterizes the individual solutions within the family) is determined by a linear homogeneous problem. This consists of the linearized flow equations (3.1) and (3.3) and the linearized balance expressions I_1, I_2 and I_3. For numerical work the functions g_i are always described by a finite number of parameters (for instance by the entries in a table for these functions). The linearized problem ultimately expresses the changes in I_1, I_2 and I_3 in terms of the changes in these parameters. For simplicity we assume that we use a collocation method. Then one requires that the functions I_1, I_2 and I_3 vanish for a number of values of ψ. One thus obtains a linear system of equations for the changes in the parameters describing the functions g_i. If for $\psi = 0$ the functions I_1, I_2 and I_3 are linearly dependent, then the system becomes degenerate, and this fact makes the existence of a solution to the homogeneous problem possible. Otherwise a neighbouring solution to a given solution of the flow problem would not exist. This is a simplified picture. The relation between changes in the functions I_i and changes in the functions g_i is given by a system of three linear integral equations. In this form the problem is investigated in appendix VI of Guderley & Greene (1967). These discussions take into account the character of the kernels, which is somewhat special in the vicinity of $\psi = 0$.

If ψ = constant then the condition $I_1 = 0$ is identically satisfied if one chooses g_1 = constant. The flow is given by a potential vortex around the axis of symmetry. It is natural to ask whether such a flow satisfies in addition the balance equations $I_2 = 0$ and $I_3 = 0$. The inhomogeneous terms in the partial differential equation (3.3) contain derivatives $dg_i/d\psi$. If the secondary flow is very slow then these derivatives must be very small. Accordingly one has for such a gas $g_2 \sim$ constant and $s = g_3 \sim$ constant. Then one finds

$$I_3 = \oint_{\psi=\text{const}} [\mu y^3 \omega(\omega_x dy - \omega_y dx) + yk(T_x dy - T_y dx)] + O(q^2).$$

For μ = constant and g_1 = constant one obtains because of (2.9)

$$\oint_{\psi=\text{const}} \mu y^3 \omega(\omega_x dy - \omega_y dx) = 2\mu g_1 \oint_{\psi=\text{const}} y^{-2} dx.$$

In the present limiting case the temperature is a function only of y. One obtains from (3.1)

$$yk(T_x dy - T_y dx) = kT_i(g_1/y^2)\, dx.$$

Therefore, under the assumption that k is constant,

$$I_3 = g_2(2\mu - kT_i) \oint_{\psi=\text{const}} y^{-2}\,dx. \tag{4.2}$$

The Prandtl number is defined by

$$Pr = (c_p\,\mu)/k. \tag{4.3}$$

For an ideal gas $T = T(i)$ and then $T_i = (\partial T/\partial i)_s = 1/c_p$. The terms in I_3 of lowest order in q^2 vanish only if $Pr = \frac{1}{2}$. A corresponding result is found for I_2. Only for $Pr = \frac{1}{2}$ can the balance conditions $I_2 = 0$ and $I_3 = 0$ be satisfied for arbitrarily slow secondary flows.

It was mentioned above that changes in the functions g_i and changes in the functions I_l are related by integral equations. It is possible to derive somewhat more detailed information which is useful for numerical work. We restrict ourselves to cases where $\mu = $ constant. Then $g_1 = $ constant and the condition $I_1 = 0$ is identically satisfied. We assume that we are in possession of a fairly close approximation to a solution of the problem. Let Δg_i and ΔI_l denote small changes in the g_i and the I_l and denote by a prime differentiation with respect to ψ. Then one has the following linearized relations:

$$\Delta I_2(\psi) = k_{11}(\psi)\,\Delta g_2'(\psi) + k_{12}(\psi)\,\Delta g_3'(\psi) + \int_0^{\psi_{\max}} K_{11}(\psi,\psi_1)\,\Delta g_2'(\psi_1)\,d\psi_1 + \int_0^{\psi_{\max}} K_{12}(\psi,\psi_1)\,\Delta g_3'(\psi_1)\,d\psi_1, \tag{4.4a}$$

$$\Delta I_3(\psi) = k_{21}(\psi)\,\Delta g_2'(\psi) + k_{22}(\psi)\,\Delta g_3'(\psi) + \int_0^{\psi_{\max}} K_{21}(\psi,\psi_1)\,\Delta g_2'(\psi_1)\,d\psi_1 + \int_0^{\psi_{\max}} K_{22}(\psi,\psi_1)\,\Delta g_3'(\psi_1)\,d\psi_1. \tag{4.4b}$$

To apply these relations one needs a fairly close approximation to the desired solution, so that a linearization can be applied. In practice we have always used as the starting approximation for a new flow field data from some other flow field for which the governing parameters are close to those of the desired flow field.

One can try to solve the above system of integral equations by means of a Neumann series. The procedure is the following. For each approximation the functions $I_2(\psi)$ and $I_3(\psi)$ are computed. The corrections to $g_2(\psi)$ and $g_3(\psi)$ must be of such a nature that $I_2(\psi) + \Delta I_2(\psi) = 0$ and $I_3(\psi) + \Delta I_3(\psi) = 0$. The left-hand sides of the above system are therefore known. In a Neumann series one disregards the contributions of the integrals and determines corrections Δg_2 and Δg_3 from the equations

$$-I_2^{(n)}(\psi) = \Delta I_2^{(n)}(\psi) = k_{11}^{(n)}(\psi)\,\Delta g_2'(\psi)^{(n+1)} + k_{12}^{(n)}(\psi)\,\Delta g_3'(\psi)^{(n+1)}, \tag{4.5a}$$

$$-I_3^{(n)}(\psi) = \Delta I_3^{(n)}(\psi) = k_{21}^{(n)}(\psi)\,\Delta g_2'(\psi)^{(n+1)} + k_{22}^{(n)}(\psi)\,\Delta g_3'(\psi)^{(n+1)} \tag{4.5b}$$

and then repeats the process. The superscripts indicate to which iteration step different quantities belong. Of course nothing can be said about the convergence of this procedure, unless one has detailed information about the kernels K_{il}. In this respect we have relied on an inspection of the results obtained in consecutive iteration steps.

To derive expressions for the functions $k_{il}(\psi)$ we study the effect of perturbations in the form of delta-functions in Δg_2 and Δg_3 on ΔI_2 and ΔI_3. It is practical to use in the derivations velocities instead of derivatives of the stream function. Let $\psi = \psi_0$ be the

streamline at which the delta-perturbations are introduced. We visualize them as a narrow layer with large gradients of g_2 and g_3. The pressure gradient in such a layer is $O(1)$. Therefore from the second law of thermodynamics

$$di/dn = T ds/dn + O(1) = T(\psi_x^2 + \psi_y^2)^{\frac{1}{2}} dg_3/d\psi + O(1), \tag{4.6a}$$

$$dT/dn = T_i\, di/dn + T_s\, ds/dn + O(1) = T(\psi_x^2 + \psi_y^2)^{\frac{1}{2}} (T_i + T_s/T)\, dg_3/d\psi, \tag{4.6b}$$

and from (2.10), together with the assumption that $g_1 = \text{constant}$,

$$d(\tfrac{1}{2}q^2)/dn = dg_2/dn - di/dn = (\psi_x^2 + \psi_y^2)^{\frac{1}{2}} (dg_2/d\psi - T dg_3/d\psi) + O(1). \tag{4.7}$$

Let $u = q\cos\theta$ and $v = q\sin\theta$. Then

$$d(\,)/dl = -\cos\theta\, \partial(\,)/\partial x - \sin\theta\, \partial(\,)/\partial y, \quad d(\,)/dn = -\sin\theta\, \partial(\,)/\partial x + \cos\theta\, \partial(\,)/\partial y,$$

$$dx/dl = -\cos\theta, \quad dy/dl = -\sin\theta.$$

Expression (2.17) can be transformed either by a detailed computation or by inspection. One then considers a point where $v = 0$, $u = q$, so that $dx = -dl$, $dy = 0$, $\partial(\,)/\partial x = -d(\,)/dl$ and $d(\,)/dy = d(\,)/dn$. One finds

$$\mu y[2(uu_x\, dy - vv_y\, dx) + (u_y + v_x)(v dy - u dx)] + ky(T_x\, dy - T_y\, dx)$$
$$= [\mu y q(dq/dn - q\, d\theta/dl) + ky\, dT/dn]\, dl.$$

If (4.4b) is valid then one has

$$\int_{\psi_0-\epsilon}^{\psi_0+\epsilon} I_3(\psi)\, dx = k_{21}(\psi_0)[g_2(\psi_0+\epsilon) - g_2(\psi_0-\epsilon)] + k_{22}(\psi_0)[g_3(\psi_0+\epsilon) - g_3(\psi_0-\epsilon)].$$

On the basis of this equation one can identify k_{21} and k_{22}. Accordingly we form

$$\int_{\psi_0-\epsilon}^{\psi_0+\epsilon} I_3(\psi)\, d\psi = \int_{\psi_0-\epsilon}^{\psi_0+\epsilon} d\psi \oint_{\psi=\text{const}} (\mu y q\, dq/dn + yk\, dT/dn)\, dl + O(\epsilon).$$

Substituting (4.6) and (4.7) one obtains

$$\int_{\psi_0-\epsilon}^{\psi_0+\epsilon} I_3(\psi)\, d\psi = \int_{\psi_0-\epsilon}^{\psi_0+\epsilon} d\psi \Big\{ (dg_2/d\psi) \oint_{\psi=\text{const}} \mu y q\, (\psi_x^2 + \psi_y^2)^{\frac{1}{2}}\, dl$$
$$+ (dg_3/d\psi) \oint_{\psi=\text{const}} y\rho T(\psi_x^2 + \psi_y^2)^{\frac{1}{2}} [-\mu + k(T_i + T_s/T)]\, dl \Big\} + O(\epsilon)$$
$$= [g_2(\psi_0+\epsilon) - g_2(\psi_0-\epsilon)] \oint_{\psi=\psi_0} \mu y q (\psi_x^2 + \psi_y^2)^{\frac{1}{2}}\, dl$$
$$+ [g_3(\psi_0+\epsilon) - g_3(\psi_0-\epsilon)] \oint_{\psi=\psi_0} y q T(\psi_x^2 + \psi_y^2)^{\frac{1}{2}} [-\mu + k(T_i + T_s/T)]\, dl.$$

Hence

$$k_{21} = \oint_{\psi=\psi_0} y q (\psi_x^2 + \psi_y^2)^{\frac{1}{2}} \mu\, dl, \tag{4.8a}$$

$$k_{22} = \oint_{\psi=\psi_0} y q T(\psi_x^2 + \psi_y^2)^{\frac{1}{2}} [-\mu + k(T_i + T_s/T)]\, dl. \tag{4.8b}$$

In the double integral occurring in the expression (2.16) for E_2, the area element $dx\,dy$ can be replaced by $-dl\,dn$. In the integrand one encounters the expression

$$T_x^2+T_y^2=(dT/dn)^2+(dT/dl)^2.$$

Even after integration with respect to dn (or rather $d\psi$) this expression is meaningless if $dT/dn\to\infty$. But as we are considering the linearized problem, a jump along the streamline $\psi=\psi_0$ is admitted only in the perturbations. For this reason the square of a delta-function does not appear. A corresponding discussion for the dissipation function Φ gives an analogous result. The integration with respect to ψ reduces the delta-function in dT/dn to a jump in T. By forming

$$\oint_{\psi_0-\epsilon}^{\psi_0+\epsilon} I_2(\psi)\,d\psi$$

one obtains for the double integral in I_2 a contribution $O(\epsilon)$. We can therefore restrict our attention to the line integral in (2.16):

$$\oint_{\psi=\text{const}}(ky/T)\,(T_x\,dy-T_y\,dx)=\oint_{\psi=\text{const}}(ky/T)\,(dT/dn)\,dl$$

$$=(dg_3/d\psi)\oint_{\psi=\text{const}} y(\psi_x^2+\psi_y^2)^{\frac{1}{2}}k(T_i+T_s/T)\,dl.$$

Hence

$$k_{11}=0,\quad k_{12}=\oint_{\psi=\psi_0} y(\psi_x^2+\psi_y^2)^{\frac{1}{2}}\,k(T_i+T_s/T)\,dl. \tag{4.9}$$

5. Thermodynamic considerations

The individual particles undergo periodic changes of state, which can be analysed as thermodynamic cycles. In this manner one can obtain some insight which goes beyond the details of a specific example. It is the strength of such discussions that one is solely concerned with temperatures and energies while details of the processes become inessential. In the inviscid flow equations no entropy change for the individual particles is permitted. In the familiar entropy–temperature graph of thermodynamics the cycle is given by a line $s=$ constant which is traversed back and forth between the temperature limits experienced by the particle. A detailed discussion would be concerned with the deviations of the cycle from this line caused by heat inputs and outputs at different temperatures. Actually we shall not go into these details. We shall determine individual contributions to the energy and entropy balances and use these to draw conclusions regarding the general nature of the cycles.

Thermodynamic discussions are based on the abstraction of quasi-equilibrium. Losses within the gas are not permissible. All heat additions (positive or negative) are assumed to come from outside reservoirs, and all work is performed by outside forces, for instance by a piston or body forces acting on the individual particles. Accordingly we assume that the gas itself is free of viscosity and heat conductivity, and replace their effects by heat inputs or outputs from outside reservoirs and by outside forces either acting on the surface of a gas volume or as body forces.

To identify different energy and entropy inputs or outputs we shall consider the balance equations for the interior of rings bounded by surfaces $\psi=$ constant. The

cycles experienced by the particles within one such ring are not identical. One obtains identical cycles (except for a shift in timing) if one considers the particles within a hollow ring bounded by surfaces ψ = constant and $\psi + d\psi$ = constant. The information for the hollow ring can be derived from that for the full rings (presumably available as function of ψ) by differentiation with respect to ψ. We have chosen to present the data for the full rings because it is these that arise during the computation and because the differentiation introduces an element of arbitrariness. In this section we describe the general ideas underlying the breakdown of the energies and entropies. Detailed derivations may be found in Guderley *et al.* (1975). All discussions are carried out under the assumption that μ = constant, from which it follows that g_1 = constant and $I_1 = 0$.

The viscosity effects play a dual role. On the one hand they cause a loss (output) of mechanical energy and on the other hand they give an input of heat energy. The formulae expressing the energy balance are (2.14) and (2.17). The second equation arises from the first by integration with respect to ψ. The terms containing the factor μ are due to viscosity. One should be aware of the fact that the physical interpretations of the integrands in the two expressions are different. The expressions R_1, R_3 multiplied by ρ express the components of the body forces due to viscosity acting on a particle. The portion $\Delta\psi \oint [(u/q) R_1 + (v/q) R_2 + \omega R_3]\, dl$ can be interpreted as the work per unit time performed by the body forces due to viscosity on the particles enclosed by surfaces ψ = constant and ψ = constant + $\Delta\psi$. To arrive at (2.17) one splits the integrand into a term which has the form of a divergence and a term which is the negative of the dissipation function. The first term is converted into a surface integral. The part of the mechanical energy (an output) that is dissipated does not appear in I_3 because it is cancelled by the input due to the heat energy of the dissipation (which is a portion of TR_4). The term in I_3 due to viscosity gives the work performed by the shear forces at the surface ψ = constant of the ring. This work must be regarded as work performed on the volume of air by outside forces. This work can be immediately separated into contributions due to the shear forces of the swirl motion and of the secondary motion. The same separation can be made in the dissipation functions; these are the expressions Φ_1 and Φ_2 defined in (2.7).

We make a further observation. The swirl motion is represented by a potential vortex around the axis of symmetry. For constant μ the potential vortex satisfies the Navier–Stokes equations. If follows that $R_3 = 0$ (and also $R_1 = R_2 = 0$). Nevertheless a decomposition of the work done by R_3 into work done at the surface of the ring ψ = constant and work dissipated in the interior is possible. For the potential vortex these two quantities cancel each other. Thus we have the following effects of the viscosity forces: input of work by shear forces at the surface ψ = constant due to the swirl flow, output (loss) of mechanical work due to the dissipation pertaining to the swirl flow (these two contributions cancel each other and do not appear in the energy balance), input of work due to the shear forces of the secondary flow, output (loss) of work due to the dissipation of the secondary flow, input of heat due to the dissipation of the primary flow and input of heat due to the dissipation of the secondary flow.

The heat input due to conduction is given by those terms in (2.14) and (2.17) which contain the factor k. The expression involving k in (2.14) appears in a form which allows immediately the application of Gauss' theorem [see (2.6)]. This input can therefore be regarded as made from outside reservoirs. Again we try to distinguish between heat

conduction due to the swirl motion and due to the secondary motion. This distinction is not directly obvious from (2.17); it is indeed somewhat artificial. In a potential vortex without a secondary flow one has a temperature distribution which is easily computed. The heat input generated by this temperature distribution is ascribed to the swirl motion. The secondary flow will modify this temperature distribution and the change of the heat input due to this modification is ascribed to the secondary flow.

The heat inputs due to the dissipation of the swirl flow and due to heat conduction ascribed to the swirl flow can be combined. One obtains a fairly simple expression which contains the factor $2-Pr^{-1}$. This is a heat input for $Pr > \frac{1}{2}$ and an output for $Pr < \frac{1}{2}$. The modification of the temperature distribution due to the secondary motion is of such a nature that this heat input is transferred to the outer wall; it cannot be arbitrarily small unless $Pr = \frac{1}{2}$.

In the entropy balance the heat energies are subdivided in the same manner, mechanical energies playing no role. We always consider the temperature history of the particles experienced in the actual flow and determine the entropy changes due to various inputs and outputs of heat. We have identified the combined heat input due to heat conduction and dissipation of the swirl flow. The entropy changes are then computed from an area integral. Again a factor $2-Pr^{-1}$ appears. We have also identified the heat input due to the dissipation of the secondary flow, thus the corresponding entropy input is readily identified. Since the entropy balance equation (2.16) is supposed to be satisfied, one finds the entropy input due to the heat conduction of the secondary flow as the negative of the sum of all other entropy inputs. For detailed formulae reference should be made to Guderley *et al.* (1975) and Guderley (1976).

6. A remark about the program

The basic algorithm has two main steps. In the first the flow field, expressed by the functions $\psi(x, y)$ and $i(x, y)$, is computed for a given shape of the swirl chamber, a given constant g_1 and given functions $g_2(\psi)$ and $g_3(\psi)$. In the second step one evaluates the functions $I_2(\psi)$ and $I_3(\psi)$. The flow computation has rather great accuracy requirements, for one needs second derivatives of ψ in the evaluation of I_2 and I_3. (In usual flow computations only first derivatives are needed.) On the other hand an efficient procedure is required, for such flow-field computations are performed very many times. Finite-difference methods did not seem well suited under these circumstances. The author has chosen an iterative method in which all terms due to compressibility are computed from results of the preceding iteration. Thus one is left with the repeated solution of Poisson's equation. This is facilitated by a conformal mapping of the interior of the swirl chamber onto the interior of the unit circle. The solution of Poisson's equation is then reduced to a Fourier analysis of the inhomogeneous term and the evaluation of integrals. In the examples shown in the present article the cross-section of the swirl chamber is a circle with radius $\frac{1}{2}$ and centre $x = 0$, $y = 1$. In this case the mapping is trivial. A few examples have demonstrated that the method is indeed effective for a non-circular contour.

In the flow field so obtained one must find contours ψ = constant by interpolation and then determine the functions $I_2(\psi)$ and $I_3(\psi)$ by integrating along these curves.

The algorithm described so far leads from the functions $g_2(\psi)$ and $(g_3(\psi)$ to the functions $I_2(\psi)$ and $I_3(\psi)$. The iteration loop is closed by means of (4.5), which deter-

mines the corrections to $g_2(\psi)$ and $g_3(\psi)$ from the functions $I_2(\psi)$ and $I_3(\psi)$. For this procedure a sufficiently close approximation to the final solution is required, but even then convergence is not guaranteed. In an earlier stage of the work a Newton–Raphson procedure was used to close the iteration loop; then closeness of the approximation will guarantee convergence. But this approach was prohibitively time consuming. A detailed description of the program and of computational experiences may be found in Guderley *et al.* (1975).†

The computations were carried out for an ideal gas with constant ratio of specific heats γ. The gas constant was R. Except for lengths all quantities were made dimensionless. Let a subscript zero denote a reference state. One then has

$$i_0/a_0^2 = 1/(\gamma-1), \quad RT_0/a_0^2 = 1/\gamma.$$

Let the actual physical quantities be temporarily characterized by a tilde. We have chosen

$$u = \tilde{u}/a_0, \quad v = \tilde{v}/a_0, \quad q = \tilde{q}/a_0, \quad \rho = \tilde{\rho}/\rho_0, \quad \psi = \tilde{\psi}/(\rho_0 a_0),$$

$$g_1(\psi) = \tilde{g}_1(\tilde{\psi})/a_0^2 = \tilde{g}_1(\psi\rho_0 a_0)/a_0^2,$$

$$g_2(\psi) = (\gamma-1)\tilde{g}_2(\psi\rho_0 a_0)/a_0^2, \quad g_3(\psi) = (\tilde{g}_3(\psi\rho_0 a_0) - s_0)/R.$$

The figures show $\Pi(\psi) \equiv \exp(-g_3(\psi))$ instead of $g_3(\psi)$.

7. Results

In the computed examples it is assumed that one is dealing with an ideal gas and that the coefficients of heat conductivity and viscosity are constant. Except for the length scale the equations are written in dimensionless form. Besides the shape of the swirl chamber one can then change the Prandtl number, the intensity of the swirl (i.e. the value of g_1) and a third parameter which characterizes the speed of the secondary motion. We have chosen for this purpose the value of $dg_2/d\psi$ at the centre of the secondary motion. Because of (4.1) this determines also $d\Pi/d\psi$, but the location of the centre of the secondary motion is not fixed in advance, therefore $d\Pi/d\psi$ may change from one iteration step to the next.

Some streamline patterns are shown in Guderley *et al.* (1975); they are rather insensitive to changes in the parameters determining the flow field. More informative are curves of $dg_2/d\psi$ and $d\Pi/d\psi$ *vs.* ψ.

Figures 2 (*a*) and (*b*) show the effect of varying Prandtl number at fixed values of g_1 and $(dg_2/d\psi)_{\psi=0}$. A qualitative understanding of the trends which appear in these figures can be obtained in the following manner. We consider as an analogue of the secondary motion a plane solid-body rotation. Then there is no secondary dissipation. One finds from (4.7)

$$d(\tfrac{1}{2}q^2)/dn = dg_2/dn - Tdg_3/dn.$$

In this model the streamlines are circles, the normal direction being that of increasing r. For a rigid-body rotation q is a predetermined function of r. By changing dg_2/dn and dg_3/dn while keeping the left-hand side constant, one changes the temperature distri-

† The author is greatly indebted to Major David Greene and Mrs Marian Valentine, who wrote and repeatedly revised his very complex program. Without their great competence and patience the work could not have been finished.

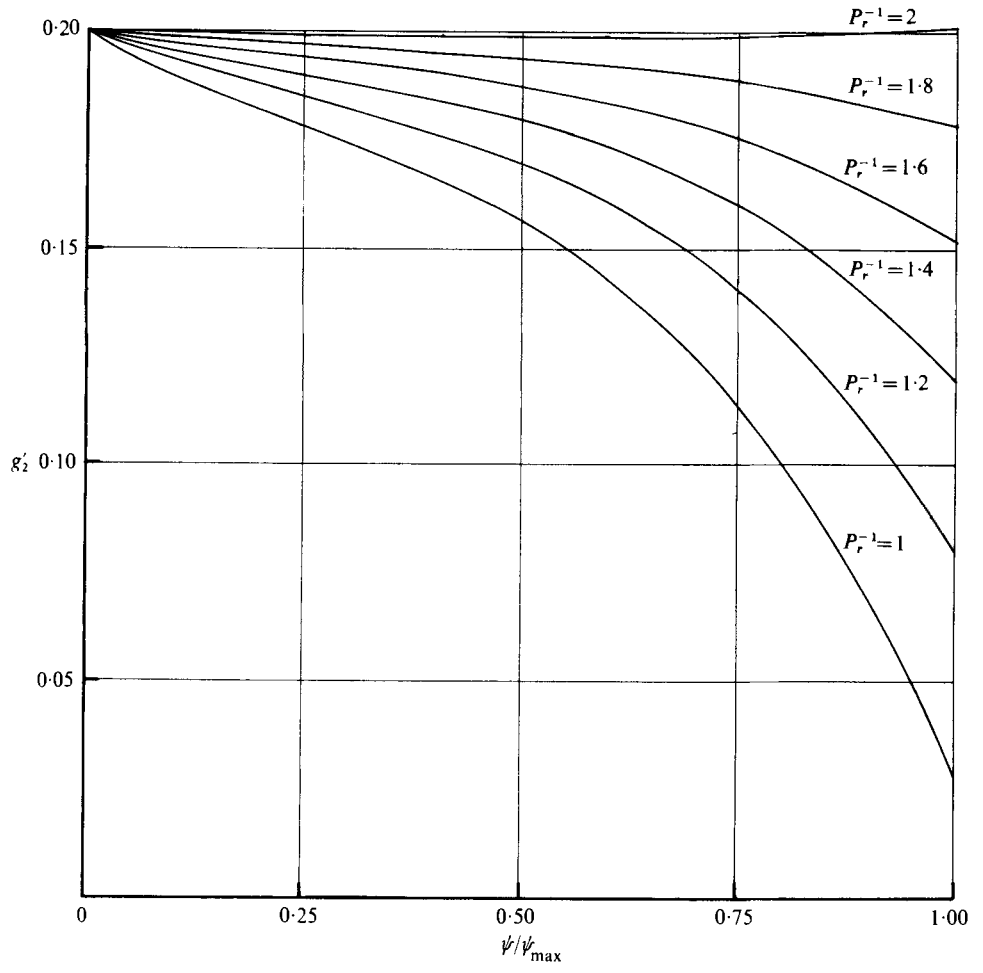


FIGURE 2(*a*). For caption see next page.

bution within the flow pattern. To obtain a temperature decrease as one moves to larger values of the radius one needs an entropy decrease, i.e. dg_3/dn and consequently also dg_2/dn must become smaller with increasing radius. For $Pr = \frac{1}{2}$ there is no heat input due to combined heat conduction and dissipation of the swirl motion and for solid-body rotation the heat input due to the dissipation of the secondary motion vanishes too. Therefore the secondary heat conduction must be zero and the secondary flow leaves the temperature distribution corresponding to the swirl flow practically unchanged. The values of $dg_2/d\psi$ and $d\Pi/d\psi$ for $Pr = \frac{1}{2}$ in figure 2 are nearly constant. As the Prandtl number is increased, the heat input due to the swirl flow is increased and then a modification of the temperature gradient due to the secondary flow must appear. The average temperature of the outside streamlines will be lowered. This explains qualitatively the Prandtl number dependence of the curves shown in figure 2.

Figures 3(*a*) and (*b*) give curves of $dg_2/d\psi$ and $d\Pi/d\psi$ *vs.* ψ for $Pr = 1$, $g_1 = 0{\cdot}6$ and different values of $dg_2/d\psi$ at $\psi = 0$ (the centre of the secondary motion). The curves terminate at the values of ψ which correspond to the boundary of the swirl chamber. Conspicuous are the large gradients of $dg_2/d\psi$ and $d\Pi/d\psi$ which appear in the vicinity

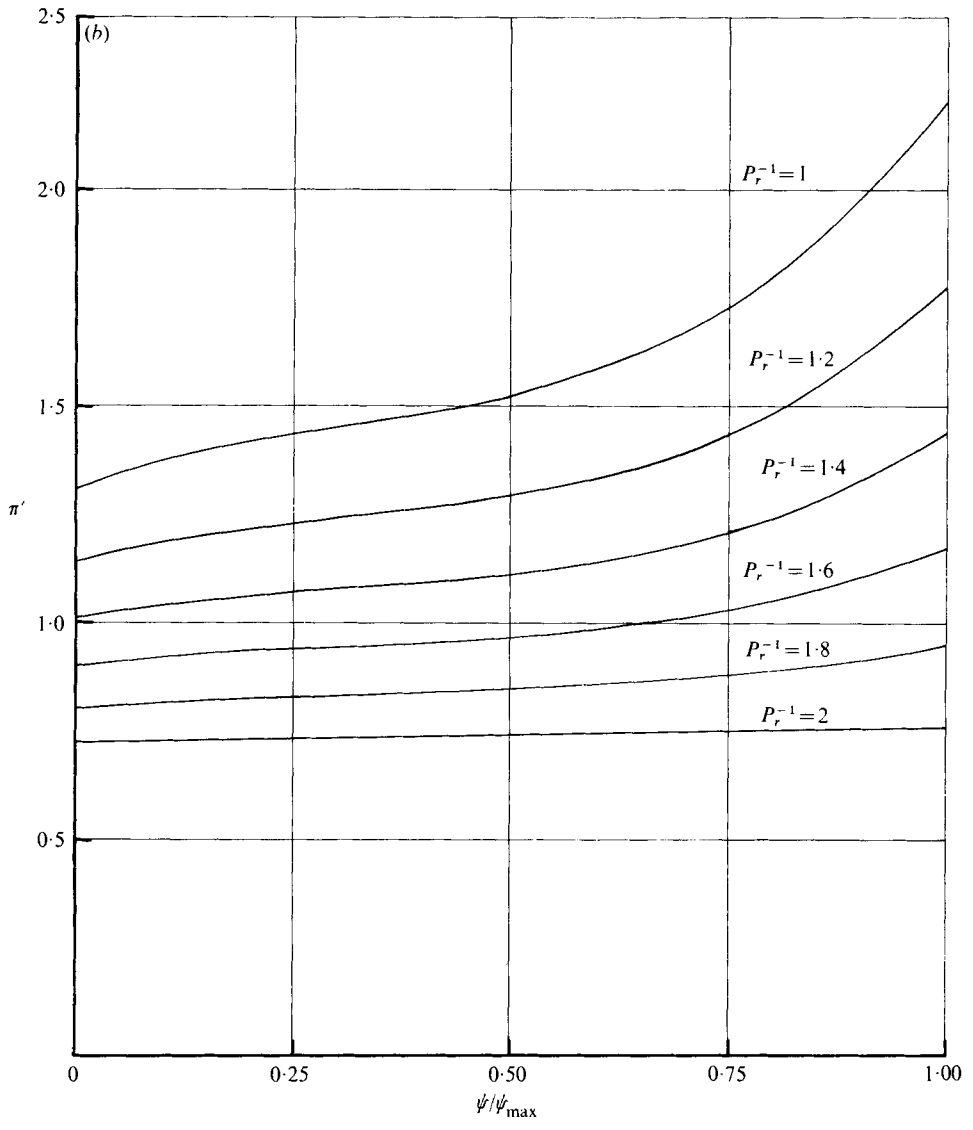


FIGURE 2. (*a*) g_2' and (*b*) Π' *vs.* ψ for different values of the Prandtl number. Circular contour, $g_1 = 0{\cdot}25$, $g_2'(0) = 0{\cdot}2$.

Pr^{-1}	2·0	1·8	1·6	1·4	1·2	1·0
$\psi_{\max}$	0·0543	0·0575	0·0613	0·0657	0·0712	0·0783

of the wall for slow secondary flows (small values of the maximal value of ψ). The trend is even more pronounced for larger values of g_1. (Such curves can be found in Guderley *et al.* 1975.) This is not a wall effect; the wall does not play a role in the present analysis and the effect is present only under special circumstances. In a rigid-body rotation, a concentrated heat addition along a circle will cause a negative jump in $dg_2/d\psi$ and $d\Pi/d\psi$. It follows that the large gradients are indicative of a large local heat input. Such a heat input can be caused by nonlinear terms in the dissipation of the secondary flow. Assume that the flow field is continued beyond the outer contour of the swirl

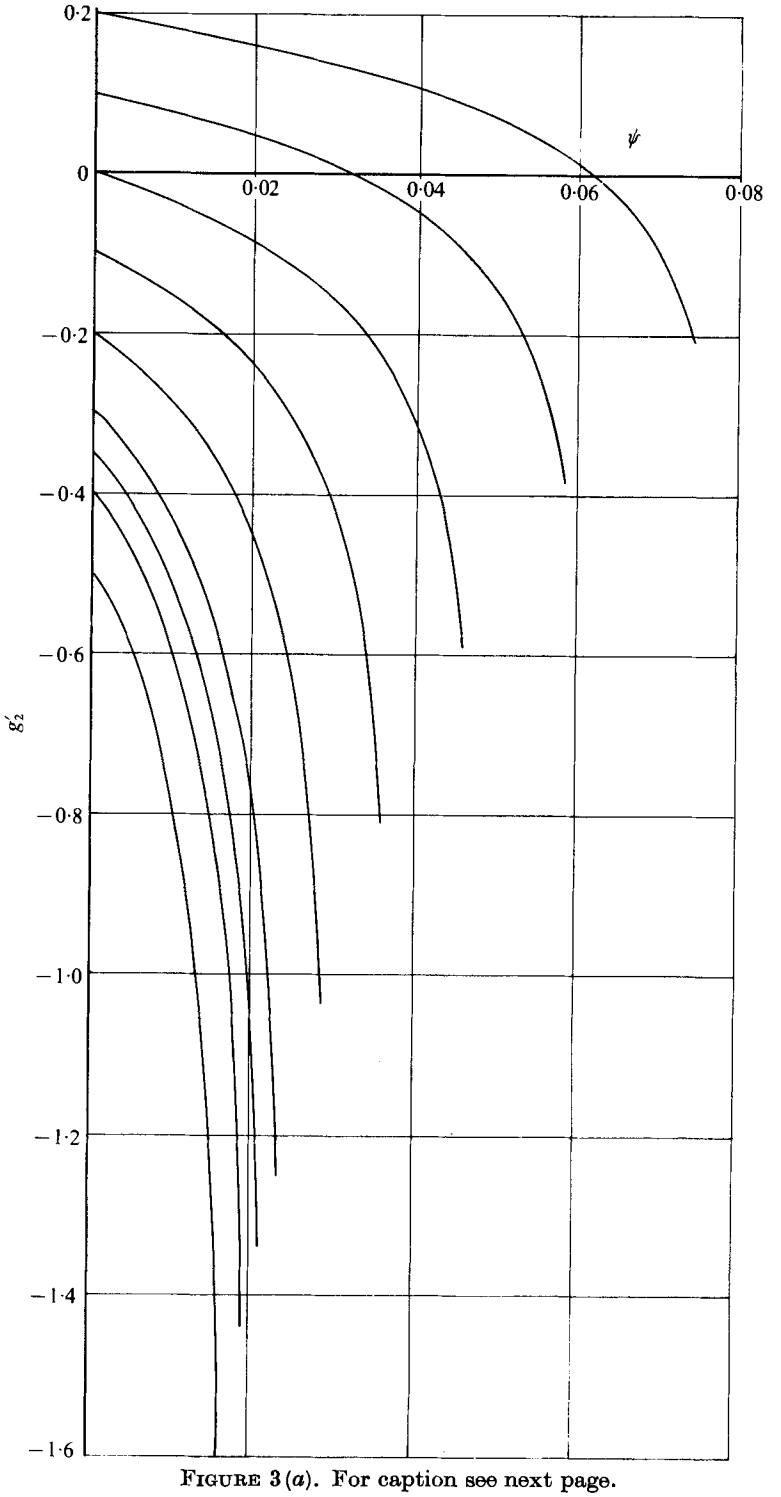


FIGURE 3 (*a*). For caption see next page.

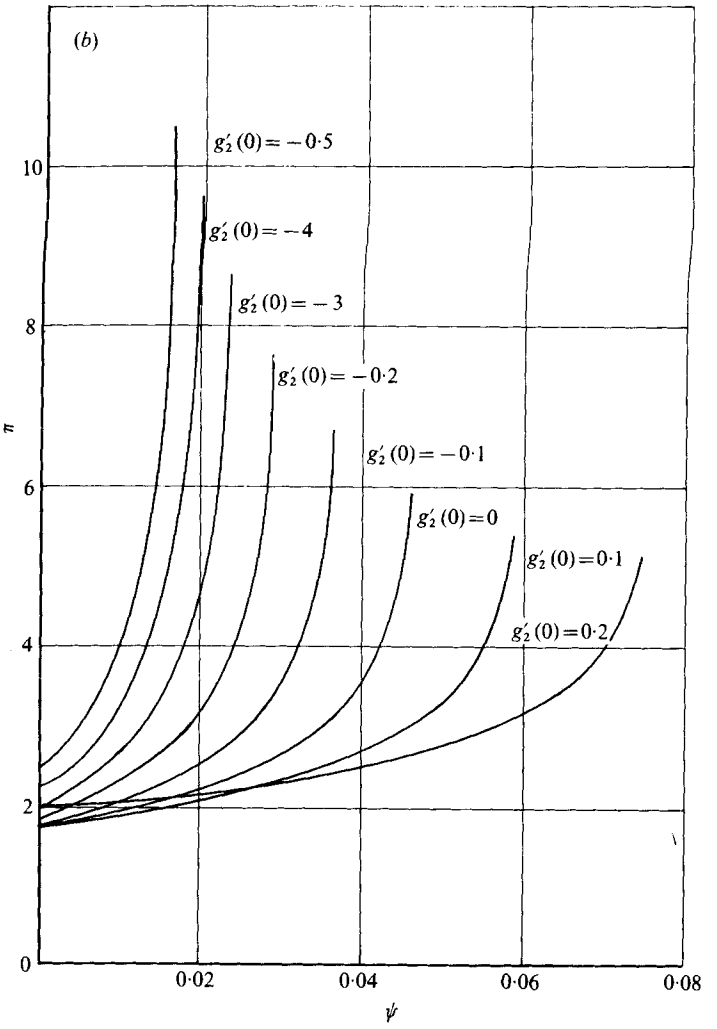


FIGURE 3. (*a*) g_2' and (*b*) Π' *vs.* ψ for different values of $g_2'(0)$. Circular contour, $Pr = 1$, $g_1 = 0{\cdot}6$.

chamber and that for a streamline $\psi = \psi_0$ which lies in this continuation $dg_2/d\psi$ and $d\Pi/d\psi$ behave as $\Delta\psi^\alpha$, where $\Delta\psi = \psi_0 - \psi$ and α is a negative constant, so far unknown Then it follows from the flow equation (3.3) that $\psi_{xx} = c_1 \Delta\psi^\alpha$ and $\psi_{yy} = c_2 \Delta\psi^\alpha$. The factors c_1 and c_2 depend upon the position of the point along the streamline $\psi = \psi_0$, but they do not vanish simultaneously. Terms of this form give contributions to the dissipation function of the form $\Delta\psi^{2\alpha}$. The subsequent integration with respect to ψ required in the expression for I_2 leads to a term of the form $\Delta\psi^{2\alpha+1}$. In the discussion in the last part of §4, these terms did no appear because there the expression had been linearized. Without a linearization these terms would have generated the square of a delta-function. In these discussions another term caused by $dg_3/d\psi$ was found to be dominant, being $O(\Delta\psi^\alpha)$. Neither of these two terms will vanish by itself as $\Delta\psi \to 0$; they must cancel each other. One therefore finds $\alpha = -1$. This explains how vertical tangents to the curves $dg_2/d\psi$ and $d\Pi/d\psi$ *vs.* ψ can come about (at a point outside the

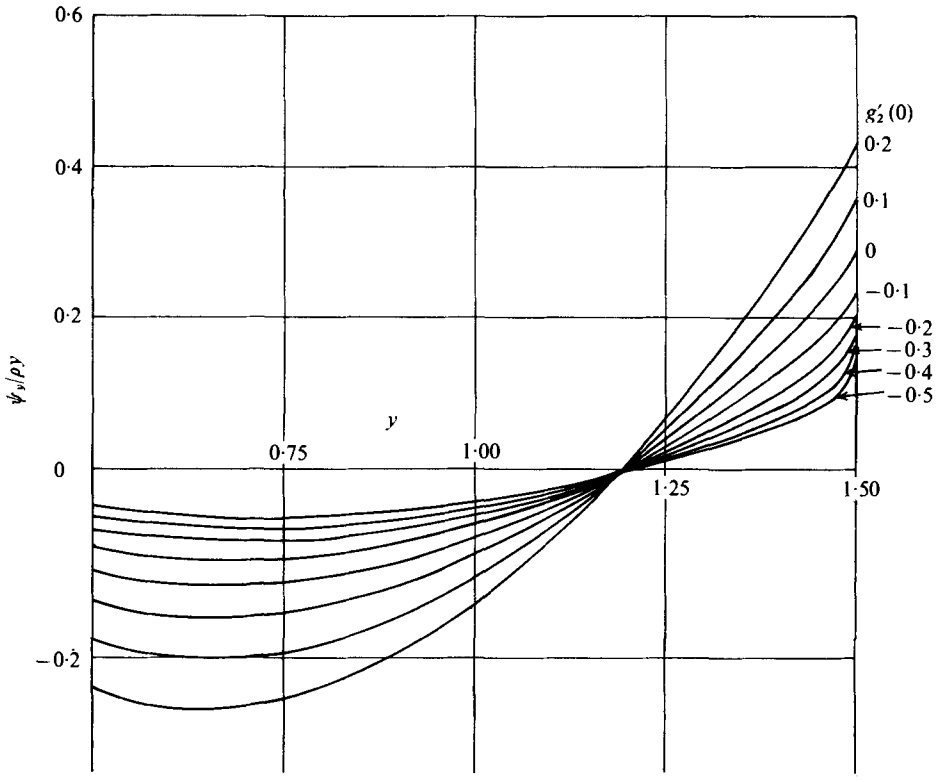


FIGURE 4. Velocity of secondary flow along the y axis for different values of $g_2'(0)$. Circular contour, $Pr = 1$, $\psi_{max} = 0{\cdot}0703$, $g_1 = 0{\cdot}6$.

swirl chamber). The steepness of the curves depends upon the closeness of this point to the contour. The explanation is not entirely satisfactory, for it implies that $g_2 \sim \log \Delta\psi$ and $\Pi \sim \log \Delta\psi$; such (weak) infinities may cause other nonlinear effects. Nevertheless the explanation makes it likely that such large heat additions are compatible with the mathematical model investigated here. One must however realize that such large local heat additions and viscosity forces violate the underlying assumptions of the present analysis. In the presence of such large gradients the effects of heat conduction and viscosity on the flow field may not be negligible. Here the value of the Reynolds number is important.

The flow-field evaluation of course gives the data needed for the computation of the boundary layer. The additional requirement that the boundary-layer streamlines be closed will ultimately determine the free parameters of the core flow. This analysis has not been carried out. The actual temperature distribution does not deviate much from the temperature distribution computed for the swirl flow alone.

Figure 4 shows the velocity of the secondary flow along the y axis for $Pr = 1$, $g_1 = 0{\cdot}6$ and different values of the parameter $(dg_2/d\psi)_{\psi=0}$. The points of intersection of the y axis with the boundary of the swirl chamber are given by $y = 1{\cdot}5$ and $y = 0{\cdot}5$. The points where the velocity goes through zero give the centres of the secondary motion. The sudden upturn of the curves at $y = 1{\cdot}5$ corresponds to the behaviour of the curves of $dg_2/d\psi$ and $d\Pi/d\psi$. This phenomenon is even more pronounced in the curves for

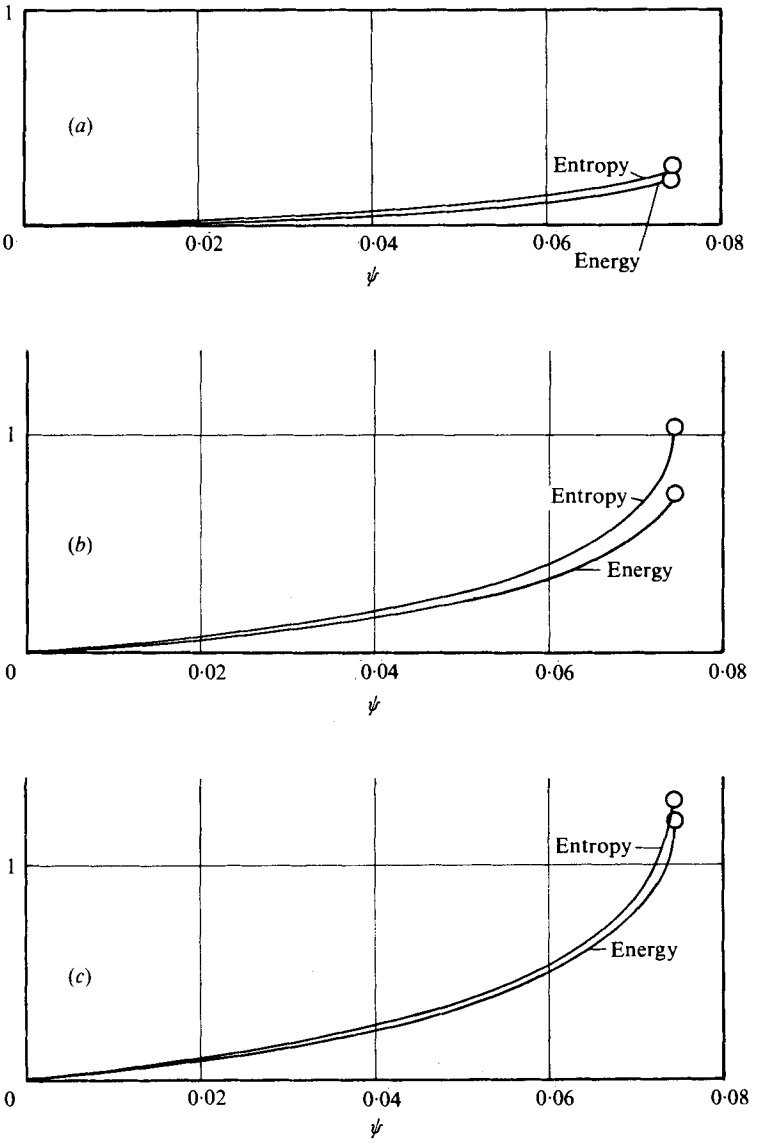


FIGURE 5. Energy and entropy inputs and outputs. Circular contour, $Pr = 1$, $g_1 = 0{\cdot}6$, $g_2'(0) = 0{\cdot}2$. (*a*) Inputs due to secondary dissipation. (*b*) Inputs due to combined dissipation and heat conduction of swirl flow. (*c*) Outputs due to secondary heat conduction. (These curves can be used to define the average temperatures for the energy inputs.)

$g_1 = 0{\cdot}8$, which can be found in Guderley *et al.* (1975) or Guderley (1976). One would expect a similar behaviour at the other boundary point, $y = 0{\cdot}5$. The data available at present do not show this behaviour, at least not in a clear-cut form. The author does not know whether this is due to a cancellation of different effects or to a lack of resolution in the numerical method. Note that these velocity distributions depend only on the choice of the functions $dg_2/d\psi$ and $d\Pi/d\psi$, not on the accuracy with which the conditions $I_2 = 0$ and $I_3 = 0$ are satisfied.

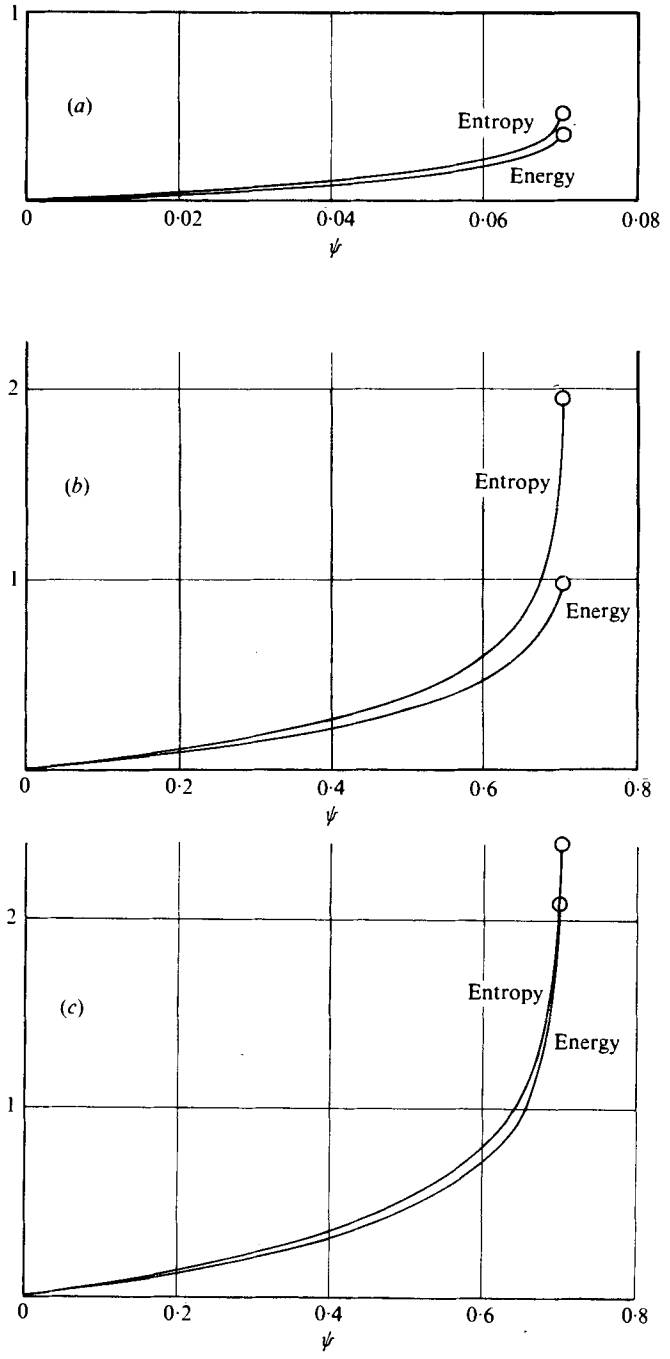


FIGURE 6. Energy and entropy inputs and outputs.
Circular contour, $Pr = 1$, $g_1 = 0{\cdot}8$, $g_2'(0) = 0{\cdot}2$. (*a*)–(*c*) as in figure 5.

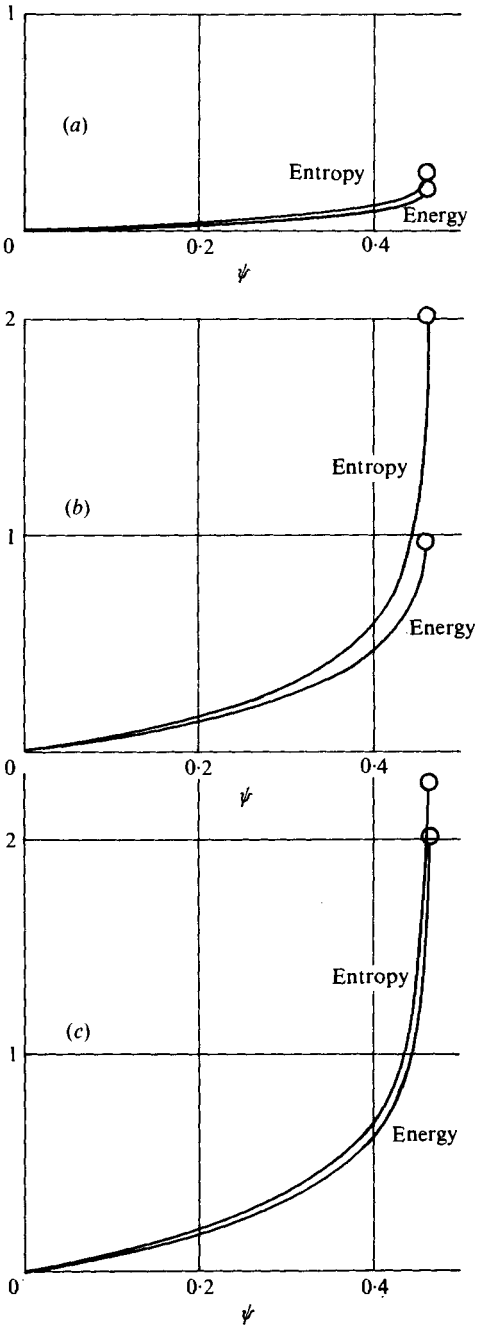


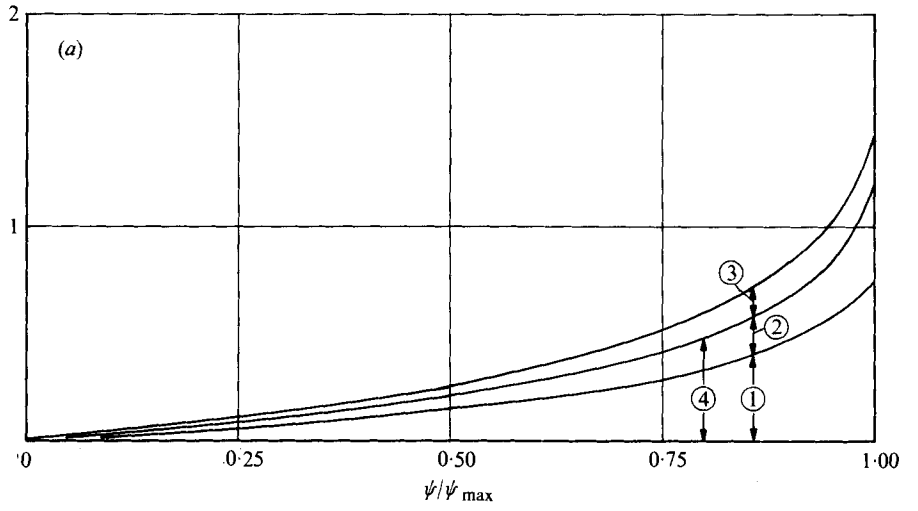
FIGURE 7. Energy and entropy inputs and outputs.
Circular contour, $Pr = 1$, $g_1 = 0{\cdot}8$, $g_2'(0) = 0$. (*a*)–(*c*) as in figure 5.

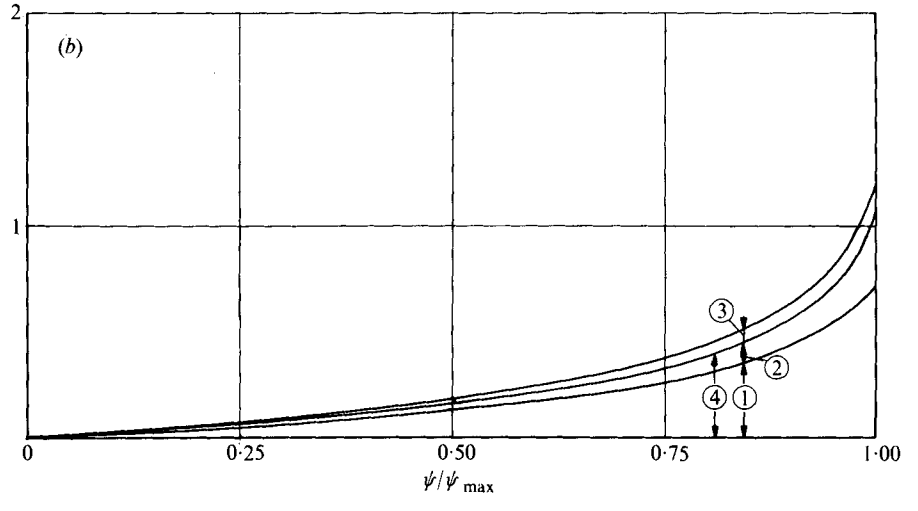
Figures 5–7 give a thermodynamic evaluation of some of the results. For the first two figures g_1 is different and the parameter $(dg_2/d\psi)$ is the same, while in the second and the third of these figures g_1 is the same and $(dg_2/d\psi)_{\psi=0}$ is different. The definitions of the different energy and entropy inputs are given in § 5. The quantities shown in the figures refer to all particles lying within a surface ψ = constant. The data for some intermediate values of $\psi = \psi_1$ are applicable to a swirl chamber which has as outer boundary the surface $\psi = \psi_1$. To obtain data for the cycles of the individual particles one must differentiate the present data with respect to ψ (this has already been explained in § 5).

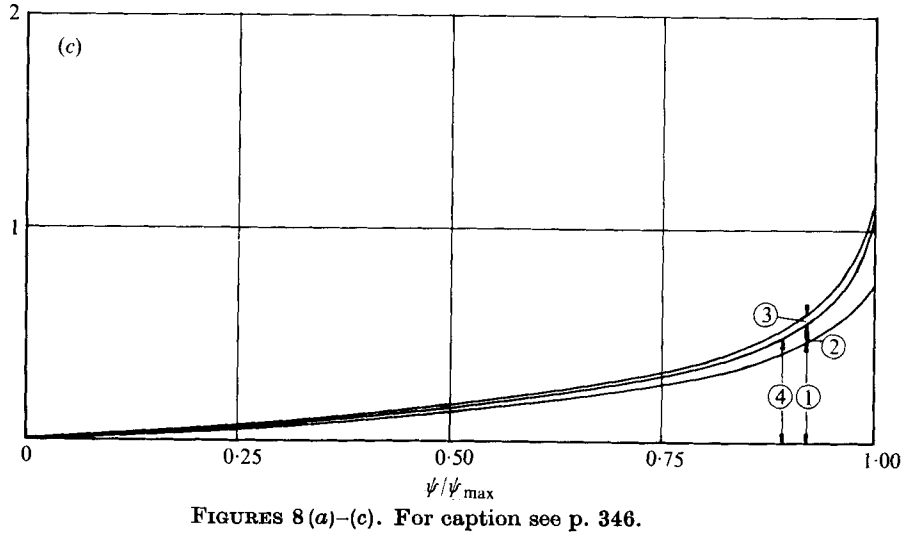
If all energy inputs occurred at a constant temperature T_0 then the energy and the entropy curves would be identical, except for a factor $1/T_0$. If the inputs occurred at a temperature lower than T_0 then the entropy curve would lie above the energy curve. The graphs show that the main heat inputs come from the combined energy dissipation and heat conduction of the swirl flow. This is the expression with the factor $2-Pr^{-1}$. Further heat input comes from the dissipation of the secondary flow. The heat conduction of the secondary flow is responsible for the heat output. The ratio between the energy and entropy curves shows the average temperatures at which these heat exchanges take place. On average the heat inputs occur at lower temperatures than the heat outputs; accordingly one needs, besides the input of mechanical energy required to maintain the swirl motion, an input of mechanical energy to maintain the secondary flow. This energy input is provided by the shear forces at the surface of the 'swirl chamber' belonging to the secondary flow. The input of mechanical energy into the secondary flow must be larger if the swirl component is larger.

Figures 8(*a*)–(*f*) show the decomposition of the energy balance for $g_1 = 0{\cdot}6$ and $g_1 = 0{\cdot}8$. Curve 1 gives the energy input due to the combined dissipation and heat conduction of the swirl motion. This portion is proportional to $2-Pr^{-1}$. Per unit area in the x, y co-ordinate system it is independent of the secondary motion; in the curves presented here one finds slight differences because of the differences in the streamline shape for the same value $\psi/\psi_{\max}$. Contribution 2 is the mechanical energy provided for the secondary flow by the shear forces at the surfaces of the fictitious swirl chambers with boundaries given by surfaces ψ = constant. Curve 3 gives the energy due to the dissipation of the secondary motion; it gives at the same time the heat input due to secondary dissipation and the output of mechanical energy, which is immediately dissipated. This portion is fairly small. Curve 4 gives the energy output due to the heat conduction of the secondary flow. This breakdown shows the relative significance of different contributions and the dependence upon the operating conditions. For a plane problem with a circular cross-section the secondary motion would be a rigid-body rotation for which the dissipation vanishes. It is therefore likely that for the swirl chamber investigated here, which also has a circular cross-section, the effect of the secondary dissipation is fairly small. In this respect the present results cannot be considered as typical.

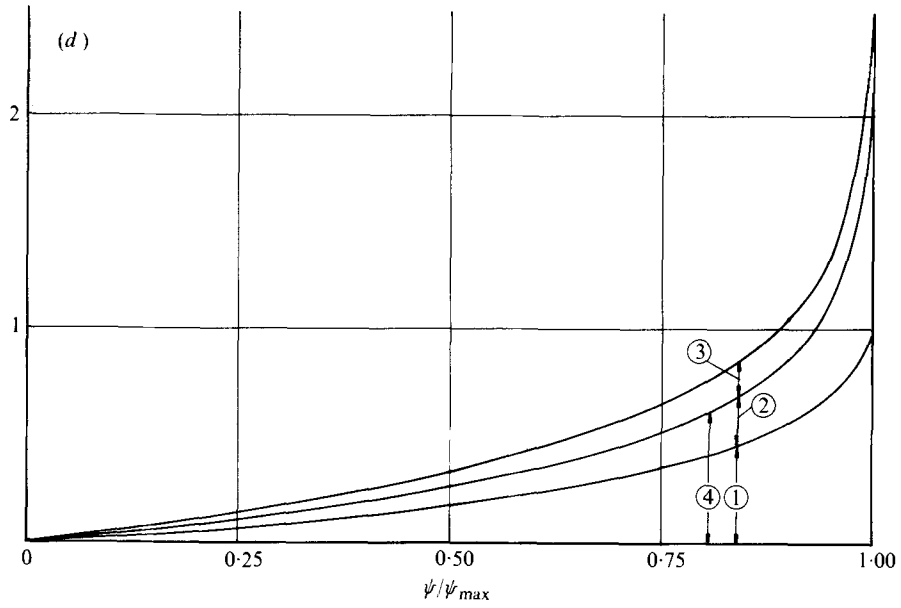
One is, of course, interested in identifying the physical mechanism which is responsible for these results. In thermodynamic discussions this mechanism does not appear. One encounters a buoyancy effect. The heat inputs occur primarily at low temperatures and low pressures, the heat outputs at high temperatures. Particles on their paths from low to high pressures have on average a higher temperature than those on the return sections of the paths. Because of this temperature difference one

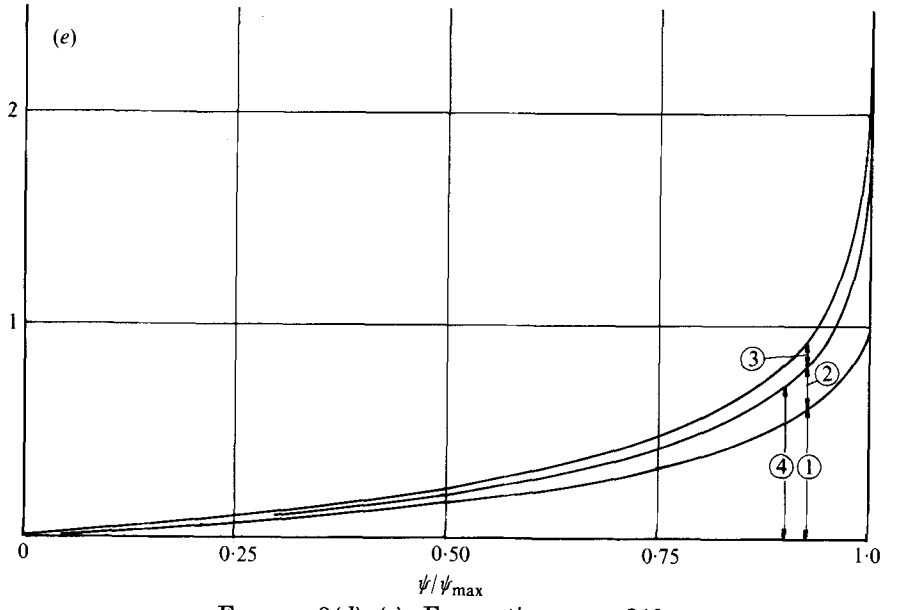






FIGURES 8 (*a*)–(*c*). For caption see p. 346.





FIGURES 8(*d*), (*e*). For caption see p. 346.

needs forces in the flow direction to maintain the secondary motion. They are supplied by the shear forces at the surfaces ψ = constant and ψ = constant + $d\psi$. The effect is more pronounced if the secondary motion is slow, for then the temperature differences will be larger.

One can draw some general conclusions. The secondary flow is rather sensitive, because the energies involved are rather small. For the case of a plane rigid rotation no energy at all is needed to maintain it. A slight heating or cooling within the flow

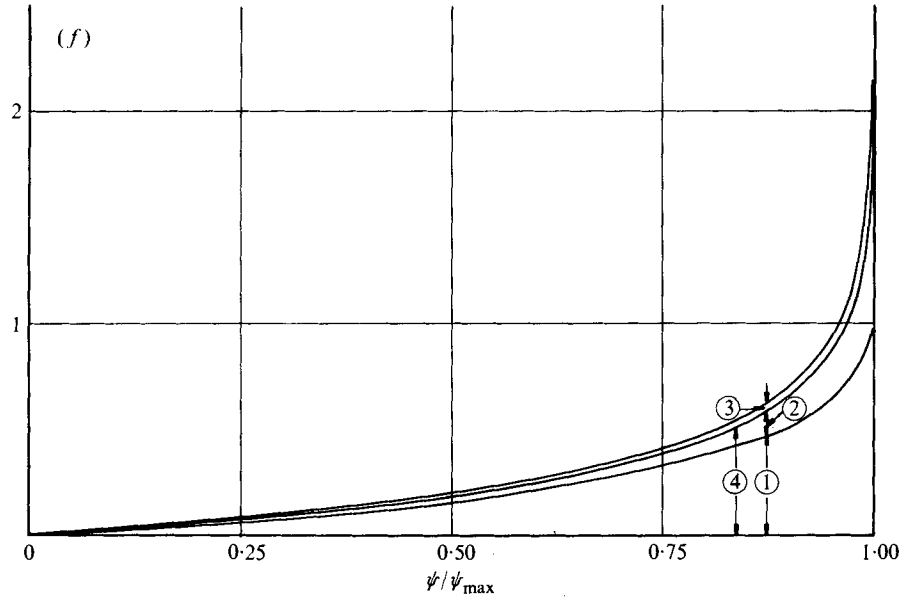


FIGURE 8. Energy decomposition *vs.* ψ. Circular contour, $Pr = 1$. ①, heat input due to swirl component (dissipation and heat conduction); ②, input of mechanical energy due to shear forces on a streamline (which defines a potential outer surface of a swirl chamber; ③, output of mechanical energy converted into heat input owing to secondary dissipation; ④, heat output due to secondary conduction.

	(*a*)	(*b*)	(*c*)	(*d*)	(*e*)	(*f*)
g_1	0·6	0·6	0·6	0·6	0·8	0·8
$g_2'(0)$	0·2	0	0·2	0·2	0	−0·2
ψ_{max}	0·0745	0·0460	0·0289	0·0703	0·0462	0·0308

field has a profound effect. In the present investigation this fact manifests itself through the dependence of the results on the Prandtl number. If heat sources were present within the flow field (for instance in the form of some slow exothermic reaction) then their effect would be similar to a change in the Prandtl number. If heat is added primarily in the low temperature portion of the cycles then it will impede the secondary flow. It may even make an orderly secondary flow of the character considered in this article impossible. Heat sources acting primarily on the high temperature portion of the cycle will favour the secondary flow.

This state of affairs implies that the results of the present investigation cannot be carried over to flow fields where one interferes with the energy balance, and that experimental results obtained for flows without heat addition cannot be readily extrapolated to flows with heat addition.

One may think of applying a finite-difference method to the Navier–Stokes equations as an alternative to the method shown in the present article. Such methods are attractive because they yield the boundary-layer flow and the core flow at the same time. The present discussions have a bearing on such methods in the following sense. Frequently finite-difference methods introduce an artificial viscosity, for instance through the very procedure of differencing. This seems to be rather innocuous in portions of the flow field where the velocity gradients are small. But we see from the present analysis that for flows with closed streamlines the energy balance is decisive. If an

artificial viscosity should appear then also a corresponding artificial heat conductivity must be present so that the effective Prandtl number is correct.

REFERENCES

BATCHELOR, G. K. 1956 On steady laminar flow with closed streamlines at large Reynolds number. *J. Fluid Mech.* **1**, 177–190.

GUDERLEY, K. G. 1976 Steady compressible swirl flows with closed streamlines at high Reynolds number. *Rep. Appl. Math. Group, Flight Dyn. Lab. (FBR), Wright–Patterson Air Force Base, Ohio.*

GUDERLEY, K. G. & GREENE, D. E. 1967 Mathematical theory of steady compressible swirl flows with closed streamlines at high Reynolds numbers. *Office Aerospace Res., U.S. Air Force. Tech. Rep. ARL* 67–0140.

GUDERLEY, K. G., GREENE, D. E. & VALENTINE, M. 1975 Computation of steady compressible swirl flows with closed streamlines at high Reynolds numbers. *Aerospace Res. Lab. LB, Air Force Systems Command Tech. Rep. ARL-TR* 75-0193.